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Functional Analysis Approach to Minimum Energy Maneuver Problem for Flexible Space Structures

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A singular optimal control problem with a performance index given by the total energy of the system is formulated, and functional analysis ideas are used to reduce the control problem to consider certain integral equations. In particular, the optimal control profile for minimum-energy control is obtained in an analytic manner. It is not necessary for the approach to take the second variation into consideration in spite of singular problems. The number of integral equations to be solved is equal to the number of control variables, which is usually much smaller than that of the state variables in the case of large space structures. The time-optimized control can be obtained numerically for the minimum-energy maneuver as the least-time maneuver that does not violate the constraints on the control inputs. The results of the present formulation are compared with that of multiple bang–bang time-optimal control using a simple model. The advantages of the present formulation are discussed on such control performance issues as the reduction of the control effort and the appropriate implementation of continuous controlled jets.

I. Introduction

THE dynamics of large space structures (LSS) feature flexible structural motions with low frequencies and very small damping ratios. To control these LSS, engineering interest is focused on performance measures such as maneuver time, control effort, and structural deformations. The time minimization problem for such control systems has been actively studied by many researchers.[1–9] For the most part, these studies are based on multiple bang–bang control with special consideration for robustness,[3,4] fuel efficiency,[5,6] and restricted deflection of flexible appendages.[9] In many cases, control costs are modeled by strictly positive quadratic terms in the cost functional, and the resulting linear, quadratic regulator formulation is widely used in control engineering. However, alternative control techniques, focused on the inherent properties of LSS systems, are of interest in meeting a variety of performance requirements.

The LSS are destined to become lightweight, extremely flexible structures with a large number of flexible modes excited very easily. The number of the actuators is usually quite limited in comparison with those of flexible vibration modes. A large number of flexible motions have no direct control force for their vibrations. Flexible motion is easily excited when LSS change their attitude or position, and it is necessary to maneuver the attitude or position of LSS with minimal excitation of structural vibration and with well-suppressed residual vibration at the final state.

In the present paper, the total energy of the system is selected as a performance index to suppress structural vibration and fuel consumption during maneuvers. Such an optimal control problem for linear dynamic systems becomes singular because the performance index does not explicitly contain the control variables. In general, a singular optimal control problem is reduced to a two-point boundary-value problem with additional second-order conditions, and it is difficult to solve the problem numerically because the number of differential equations to be managed increases in accordance with the size of increases of the state variables. To avoid this numerical difficulty, an alternative analytical procedure is employed as a new approach to singular optimal control problems in the present paper.

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In the problem formulation, it is shown that a singular optimal control problem can be reduced to the analysis of certain integral equations by using a nonlinear programming procedure, and the optimal control profile is obtained in an analytic manner. It is also shown that the second-order conditions are not necessary in the present formulation because the functional to be minimized is convex. The present approach is distinguished from other studies[1–10] in the respect that the functional to be minimized includes integral-formed control variables. The present method is useful for systems such as LSS in which the dimension of the control variables is much smaller than that of the state variables. This is because the number of integral equations to be solved is the same as the number of controllers available in the system, which is usually small in comparison to the number of degrees of freedom to be controlled in the state variables of the LSS.

To illustrate the present method, an example treats a simple model consisting of one rigid mode and one flexible mode actuated linearly in the horizontal plane. The problem is to transfer the position of the system from an equilibrium state with no transitional velocity and flexural vibration (rest state) to another equilibrium state with no velocity and flexural vibration (rest state). Minimum-time maneuvers are studied within this family of energy-minimizing control, and the time-optimized control can be obtained numerically as the least-time maneuver that does not violate the constraints on the control inputs. The resulting control profile is compared with that of multiple bang–bang control for time minimization control to confirm the validity and advantage of the present formulation.

The present formulation of the minimum-energy, minimum-time optimal control shows some interesting results in comparison with the usual standard treatment of the time-optimal control problem employed with the multiple bang–bang control. First, the time-optimal control in the bang–bang control eventually results in multi-switching bang–bang control, which is adequate for an on–off actuator such as a pulse jet. The present problem formulation, however, results in a smooth input profile suitable for a continuous actuator such as an analog-controlled jet. Second, as a result of minimization of the total energy in the system, fuel is saved with little sacrifice of the control time.

II. Problem Formulation

The present problem is formulated as a rest-to-rest maneuver problem with minimum-energy in the control of a linearized dynamic system with one rigid and n vibration modes and is stated as follows:

Minimize

$$J = \frac{1}{2}\int_0^{t_f} x^T(t)Qx(t)\,\mathrm{d}t \tag{1}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2}$$

$$x(0) = x_0, \qquad x(t_f) = x_f \tag{3}$$

$$|u_j(t)| \le 1 \qquad (j = 1, 2, \ldots, m) \tag{4}$$

where t_f is the specified final time and

$$A = \begin{bmatrix} 0 & 1 & & & & & 0 & \\ 0 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -\omega_1^2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ & 0 & & & & & -\omega_n^2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & & 0 \\ \phi_{01} & & \phi_{0m} \\ 0 & & 0 \\ \phi_{11} & \cdots & \phi_{1m} \\ \vdots & & \vdots \\ 0 & & 0 \\ \phi_{n1} & & \phi_{nm} \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & & & & & 0 & \\ 0 & 0 & & & & & & \\ & & \omega_1^2 & 0 & & & & \\ & & 0 & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & & \omega_n^2 & 0 \\ 0 & & & & & & 0 & 1 \end{bmatrix}$$

and $x \in H_1$ and $u \in H_2$ with H_1 and H_2 the usual Hilbert spaces spanned by $2(n+1)$-dimensional and m-dimensional square integrable functions, respectively, and inner products are defined as follows for $x,\ y \in H_i$, $i = 1, 2$:

$$\langle x, y\rangle_i = \int_0^{t_f} x^T(t)y(t)\,\mathrm{d}t$$

The performance index (1) is proportional to the time-averaged sum of the kinetic and the elastic energy of the flexible modes, and minimizing the performance index means suppression of the vibration motion of flexible appendages of the LSS. The solution of the state equation (2) is obtained under the boundary conditions (3) as follows:

$$x(t) = \int_0^t \exp[A(t-\tau)]Bu(\tau)\,\mathrm{d}\tau := (Lu)(t) \tag{5}$$

$$(Lu)(t_f) = x_f \tag{6}$$

where L is a linear integral operator and zero initial conditions, $x(0) = 0$, are assumed for the present rest-to-rest maneuver. The constraints on the control variables (4) are alternatively expressed as follows:

$$\varphi[u(t)] = \begin{bmatrix} 1 - u_1^2(t) \\ 1 - u_2^2(t) \\ \vdots \\ 1 - u_m^2(t) \end{bmatrix}, \qquad \varphi_j[u(t)] = 1 - u_j^2(t) \ge 0 \tag{7}$$

The Lagrangian $I(u, \lambda)$ is defined as

$$\begin{aligned} I(u,\lambda) &= \nu^T[x(t_f) - x_f] + J - \int_0^{t_f} \lambda^T(t)\varphi(u)\,\mathrm{d}t \\ &= \nu^T[(Lu)(t_f) - x_f] + \frac{1}{2}\langle Lu, QLu\rangle_1 - \langle \lambda(t), \varphi(u)\rangle_2 \end{aligned} \tag{8}$$

where $\nu \in R^{2(n+1)}$, $\lambda, \varphi \in H_2$, and the Lagrangian are explicitly expressed in Eq. (8) as a form of the terms of the control variables u and without the state variables x. The original problem described by Eqs. (1–4) with time derivatives is transformed into a nonlinear programming problem without time derivatives. The objective is to find an optimal control profile to minimize the performance index (1), and it is interpreted as finding an optimal control function that minimizes the Lagrangian (8), with respect to the control variables. Note that it is guaranteed that the Lagrangian is convex with respect to the control variables; then the optimal control function exists. Moreover, this indicates that the second-order conditions are not necessary, although the original performance index (1) does not contain any control variables terms.[11] Refer to the Appendix for justification of the convexity claim.

From the Kuhn–Tucker theorem (see Ref. 12), the following necessary conditions for the optimal control are obtained for $j = 1, 2, \ldots, m$:

$$\lambda_j(t)[1 - u_j(t)] = 0, \qquad \lambda_j \geq 0, \qquad 1 - u_j(t) \geq 0 \tag{9}$$

$$B^T \exp\left[A^T(t_f - t)\right]\nu + L^* Q L u(t) + 2\Lambda(t)u(t) = 0 \tag{10}$$

where L^* is a conjugate operator of L,

$$\Lambda(t) = \begin{bmatrix} \lambda_1(t) & & & 0 \\ & \lambda_2(t) & & \\ & & \ddots & \\ 0 & & & \lambda_m(t) \end{bmatrix}$$

$$L^* Q L u(t) = \int_0^{t_f} Y(t, \tau) u(\tau)\, \mathrm{d}\tau \tag{11}$$

$$Y(t, \tau) = \int_{\max(t,\tau)}^{t_f} B^T \exp[A^T(s - t)] Q \exp[A(s - \tau)] B\, \mathrm{d}s \tag{12}$$

From Eq. (9), the relationship between the jth components of u and λ is expressed as follows.

The bang–bang mode:

$$\lambda_j > 0, \qquad u_j = \pm 1 \tag{13}$$

The singular mode:

$$\lambda_j = 0, \qquad |u_j| < 1 \tag{14}$$

for $j = 1, 2, \ldots, m$. The optimal control profile is obtained in the singular mode, and the Lagrange multiplier λ is to be solved in the bang–bang mode. In the singular mode, Eq. (10) becomes a Fredholm integral equation of the first kind. It is difficult to solve the Fredholm integral equation in general; however, in the case when the system is represented by a linear system as in the present study, the Fredholm integral equation can be converted to the Volterra integral equation of the second kind, which can be easily solved. In the case where all control inputs are in the singular mode, that is, $\lambda_j = 0$ and $|u_j| < 1$ for all $j = 1, 2, \ldots, m$, Eq. (10) is written as follows:

$$\int_0^t Y_1(t, \tau) u(\tau)\, \mathrm{d}\tau + \int_t^{t_f} Y_2(t, \tau) u(\tau)\, \mathrm{d}\tau = f(t)$$

$$f(t) = -B^T \exp\left[A^T(t_f - t)\right]\nu \tag{15}$$

where, for a given final time t_f, $\max(t, \tau) = t$, that is,

$$t > \tau, \qquad Y_1(t, \tau) = \int_t^{t_f} B^T \exp[A^T(s - t)] Q \exp[A(s - \tau)] B\, \mathrm{d}s$$

and $\max(t, \tau) = \tau$, that is,

$$t < \tau, \qquad Y_2(t, \tau) = \int_\tau^{t_f} B^T \exp[A^T(s - t)] Q \exp[A(s - \tau)] B\, \mathrm{d}s$$

and $Y_1(t, \tau)$ and $Y_2(t, \tau)$ are expressed as follows:

$$Y_1(t, \tau) = (t_f - t) B^T D(t, \tau) B$$

$$Y_2(t, \tau) = (t_f - \tau) B^T D(t, \tau) B$$

$$D(t, \tau) = \begin{bmatrix} D_0(t, \tau) & & & 0 \\ & D_1(t, \tau) & & \\ & & \ddots & \\ 0 & & & D_n(t, \tau) \end{bmatrix}$$

$$D_i(t, \tau) = \begin{bmatrix} \omega_i^2 \cos\omega_i(t - \tau) & \omega_i \sin\omega_i(t - \tau) \\ -\omega_i \sin\omega_i(t - \tau) & \cos\omega_i(t - \tau) \end{bmatrix}$$

Where both sides are differentiated with respect to time t twice, integral equation (15) is expressed as follows:

$$k_1 u(t) + \int_0^{t_f} k_2(t, \tau) u(\tau)\, \mathrm{d}\tau = g(t) \tag{16}$$

where

$$k_1 = -B^T B$$

$$k_2(t, \tau) = -B^T \left\{ 2\frac{\partial}{\partial t} D(t, \tau) + (t - \tau)\frac{\partial^2}{\partial t^2} D(t, \tau) \right\} B$$

$$g(t) = \begin{bmatrix} \sum_{i=0}^{n} (a_{i1} \sin\omega_i t + b_{i1} \cos\omega_i t) \\ \vdots \\ \sum_{i=0}^{n} (a_{im} \sin\omega_i t + b_{im} \cos\omega_i t) \end{bmatrix}$$

and a_{ij} and b_{ij} are unspecified constants, as will be explained later in this section.

Equation (16) can be solved by using Laplace transforms because the kernel $k_2(t, \tau)$ is expressed as $k_2(t - \tau)$. The Laplace transform of Eq. (16) becomes

$$U(s) = \{k_1 + K_2(s)\}^{-1} G(s) \tag{17}$$

where $G(s)$ and $K_2(s)$ are the Laplace transforms of $g(t)$ and $k_2(t, \tau)$, respectively. Finally, the control profiles are obtained in the following form:

$$u_j(t) = \sum_{i=1}^{n} [q_{ij1} \exp(\alpha_{ij} t) \sin\beta_{ij} t + q_{ij2} \exp(-\alpha_{ij} t) \sin\beta_{ij} t$$
$$+ q_{ij3} \exp(\alpha_{ij} t) \cos\beta_{ij} t + q_{ij4} \exp(-\alpha_{ij} t) \cos\beta_{ij} t] \tag{18}$$

for $j = 1, 2, \ldots, m$. Each α_{ij} (β_{ij}) in Eq. (18) is the real part (imaginary part) of the characteristic roots of the right-hand side of Eq. (17), respectively, and thus is determined by the denominators of Eq. (17). The coefficients $q_{ij1}, \ldots, q_{ij4}$ are constants determined by the boundary conditions. The multiplier ν in Eq. (15) and constants a_{ij} and b_{ij} in $g(t)$ need not be determined because these appear only in the numerators of Eq. (17). It is easily seen from the control profiles in Eq. (18) that the present algorithm is appropriate for implementation of such analog actuators as analog-controlled jets in comparison with the control profile of the multiple bang–bang control suffice to be implemented for the pulse jets.

III. Numerical Simulations

In this section the preceding analysis procedure is applied to illustrate the optimal control profiles.

The model of the LSS employed in this study is shown schematically in Fig. 1. The model is a rigid body equipped with a flexible beam and is actuated to move linearly along the horizontal line. An actuator is placed on the rigid body, but no control force is available for the flexible motion of the beam.

The equations of motion are obtained as follows:

$$m_0 \ddot{y}_0(t) = u(t) - \int_0^L \rho_1 \left\{ \ddot{y}_0(t) + \frac{\partial^2 y_1(x_1, t)}{\partial t^2} \right\} \mathrm{d}x_1 \tag{19}$$

$$EI \frac{\partial^4 y_1(x_1, t)}{\partial x_1^4} + \rho_1 \frac{\partial^2 y_1(x_1, t)}{\partial t^2} = -\rho_1 \ddot{y}_0(t) \tag{20}$$

$$y_0(0) = y_1(x_1, 0) = \dot{y}_0(0) = \dot{y}_1(x_1, 0) = 0 \tag{21}$$

$$y_0(t_f) = y_f, \qquad y_1(x_1, t_f) = \dot{y}_0(t_f) = \dot{y}_1(x_1, t_f) = 0 \tag{22}$$

where m_0 is the mass of the main body, $y_0(t)$ and $y_1(x_1, t)$ are the position of the main body and the deformation of the beam at

Table 1 Model parameters

Parameter	Value
Beam	
Material	Aluminum
Width	25.0 mm
Height	1.95 mm
Length	0.865 mm
Main body	
Mass	14.0 kg
Final position	5.0 m
Control input	
Final time	
t_f	4.2 s
Switching time	
t_1	0.3 s
t_2	3.9 s
Saturation force	50 N

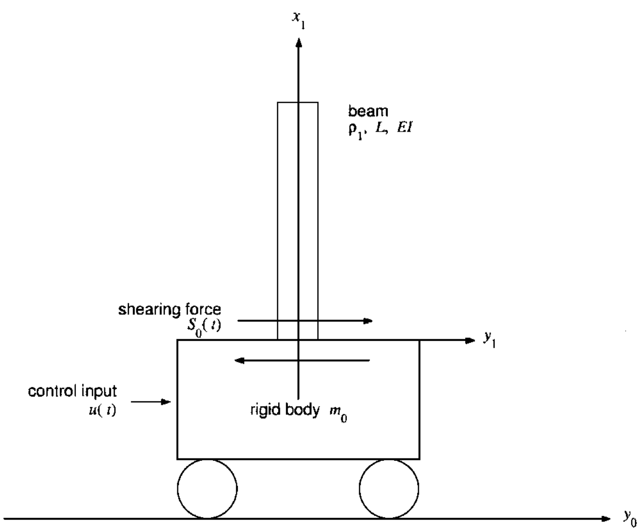

Fig. 1 LSS model.

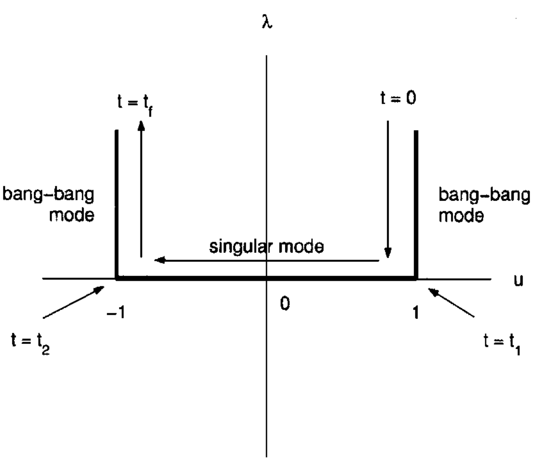

Fig. 2 Relationship between u and λ.

x_1, respectively, and ρ_1, L, and EI are the mass per unit length, the length, and the bending stiffness of the beam, respectively. The beam is assumed to be an Euler–Bernoulli beam. The equations of motion are orthogonalized with use of the modal model as follows:

$$\ddot{\eta}_0(t) = \phi_0 u(t) \tag{23}$$

$$\ddot{\eta}_1(t) + \omega_1^2 \eta_1(t) = \phi_1 u(t) \tag{24}$$

where the second and higher modes of vibration are truncated to simplify the analysis.

The minimum-energy maneuver problem is analyzed numerically for the model with the parameters shown in Table 1. In the present study, the number of switching times is selected to be two, t_1 and t_2. The relationship between the control variable u and the Lagrange multiplier λ is interpreted by Eqs. (13) and (14) as shown in Fig. 2. One of the natural paths is a one-way path starting from the bang–bang mode going to the singular mode and then entering into the bang–bang mode as indicated by arrows in Fig. 2. Thus,

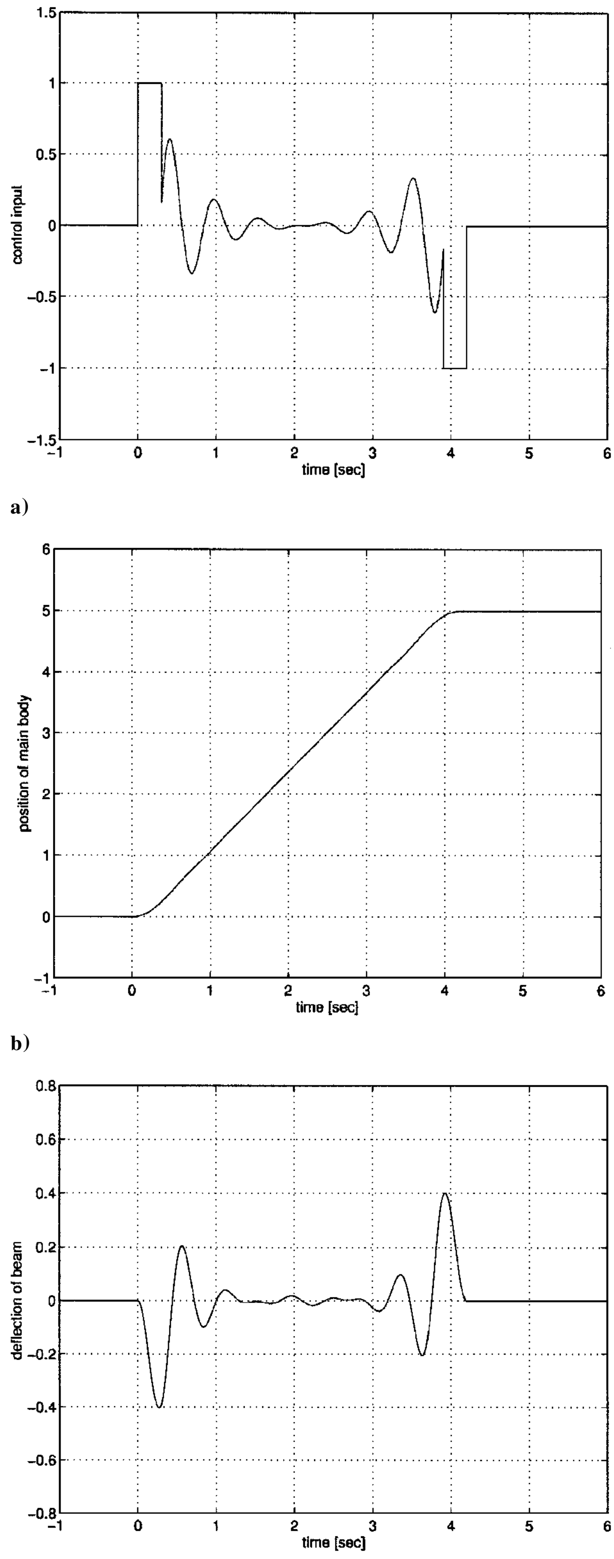

Fig. 3 Control profile and time responses of the model (t_1 = 0.30 s and t_f = 4.20 s).

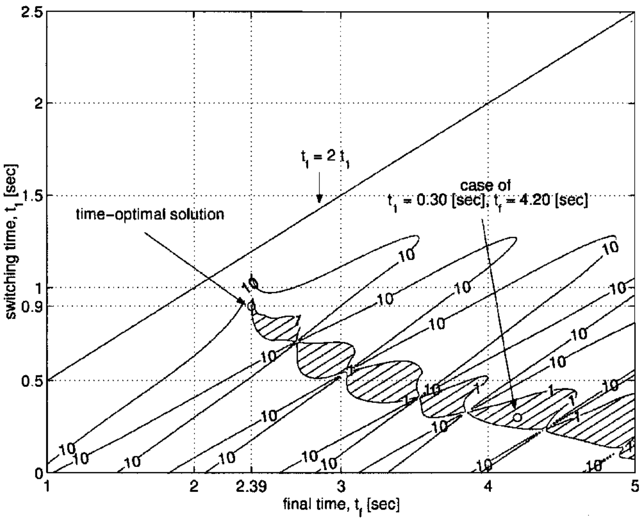

Fig. 4 Feasible region satisfying $|u| \leq 1$ (shaded area).

switching time t_1 is the time when the control profile of the bang–bang mode changes into that of the singular mode, and switching time t_2 is the control profile of the singular mode changing into that of bang–bang mode. Moreover, the switching times are assumed to be symmetric with respect to the midmaneuver time, that is, $t_2 = t_f - t_1$, and this assumption is applied in the rest of the paper. These switching times have an important role in the minimization of the final time, as will be shown in the next section. The results of the numerical simulation are shown in Fig. 3 as the control profile (Fig. 3a), the time responses of the main body position (Fig. 3b), and the amplitude of the tip of the beam (Fig. 3c). The control profile varies from $u = +1$, then to singular, and finally to $u = -1$ in sequence.

IV. Time Optimization

In this section minimum-time maneuvers are studied within this family of energy minimizing controls. The optimal final and switching times, t_f and t_1 (and $t_2 = t_f - t_1$), are not yet specified in the minimum-energy maneuver, and it is natural to seek the time-optimal solution minimizing the possible t_f because the shortest time may be an important criterion. The time-optimal minimum-energy control is studied numerically to minimize the maneuver time based on the preceding analysis through the selection of the final and switching times, t_f and t_1.

There exist situations when the values of the control input in the singular mode exceeds the prescribed maximum value of the control input.[13] Thus, it is necessary to ensure that the obtained control profile satisfies the constraint $|u| \leq 1$. The situation is shown in Fig. 4, which is a contour graph obtained by calculating the maximum value of $|u|$ of Eq. (18) associated with the variation of the final time t_f and the switching time t_1. The shaded area in Fig. 4 denotes the region where the constraint is satisfied. Additional lines denote contours $|u| = 10$, which are not feasible. The line, $t_f = 2t_1$, denotes the case of simple bang–bang control with $u = 1$ for $0 \leq t \leq t_1 = t_f/2$ and $u = -1$ for $t_2 = t_1 \leq t \leq t_f$. It is easily understood that if the bang–bang mode with $u = 1$ or $u = -1$ does not exist, that is, $t_1 = 0$, then no feasible region (shaded area) exists in Fig. 4, which for $t_f < 5$ s. The existence of the switching times, that is, $t_1 > 0$, enables the existence of the feasible region and, thus, enables the decrease of the time of maneuver t_f. At the extremum, the minimum-time solution under the present solution structure (bang–singular–bang) is easily seen to be obtained as $t_f = 2.39$ s and $t_1 = 0.90$ s from Fig. 4. This time-optimal behavior is shown in Fig. 5 as the control profile (Fig. 5a), the time responses of the main body position (Fig. 5b), and the amplitude of the tip of the beam (Fig. 5c) at the minimum-time solution. The maximum amplitude of the beam in the minimum-time case is seen to be the same as that in the case when $t_f = 4.20$ s.

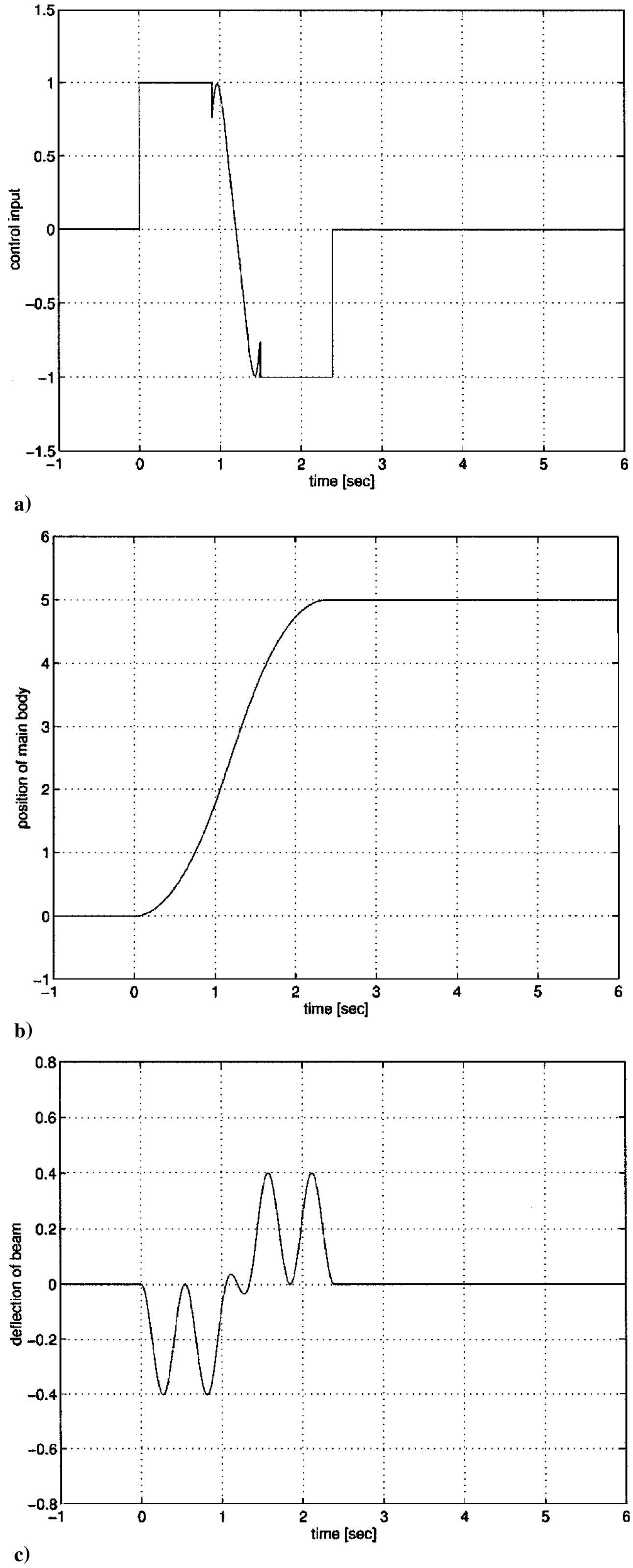

Fig. 5 Control profile and time responses of the model (t_1 = 0.90 s and t_f = 2.39 s).

V. Comparison with Multiple Bang–Bang Control

The minimum-energy, minimum-time optimized maneuver in the preceding section is compared with the multiple bang–bang control to study the performance of the proposed control profile. As far as the nth mode of vibration is concerned, the number of switching times of multiple bang–bang control is equal to $2n + 1$, in most cases.

The time-optimal control problem is reduced to the following parameter optimization problem in the multiple bang–bang control formulation[4]:

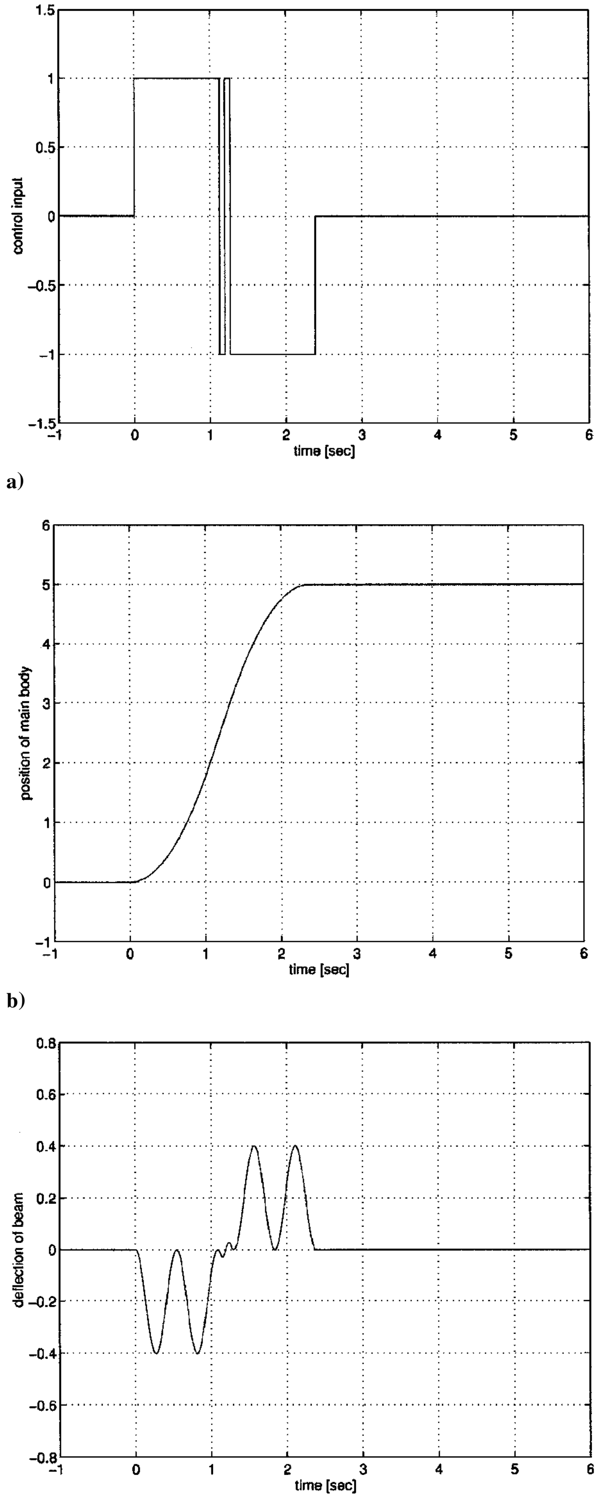

Fig. 6 Time-optimal control profile and time responses of the model (t_1 = 1.1204 s, t_2 = 1.1923 s, t_3 = 1.2643 s, and t_f = 2.3847 s).

Minimize the final time t_f such that

$$\sum_{i=1}^{2n+1}(-1)^i(t_f - t_i)^2 + \frac{t_f^2}{2} - \frac{x_{f1}}{\phi_0} = 0 \tag{25}$$

$$2\sum_{i=1}^{2n+1}(-1)^i \cos\omega_j(t_f - t_i) + \cos\omega_j t_f + 1 = 0 \tag{26}$$

$$2\sum_{i=1}^{2n+1}(-1)^i \sin\omega_j(t_f - t_i) + \sin\omega_j t_f = 0 \tag{27}$$

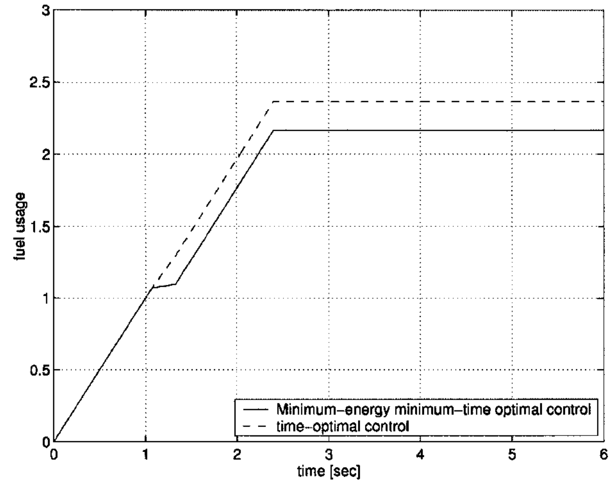

Fig. 7 Time histories of fuel usage.

for $j = 1, 2, \ldots, n$, where $t_i, i = 1, 2, \ldots, 2n+1$, are the switching times for the multiple bang–bang control and n is the number of flexible modes of the system to be retained.

In the present study, only the first mode of vibration is considered, and the switching times to be determined are t_1, t_2, and t_3 in the multiple bang–bang control. The switching times are obtained as $t_1 = 1.1204$ s, $t_2 = 1.1923$ s, $t_3 = 1.2643$ s, and $t_f = 2.3847$ s for the same system parameters as in the preceding section. Figure 6 shows the time-optimal control profile (Fig. 6a), time responses of the position of main body (Fig. 6b), and the amplitude of the tip of the beam (Fig. 6c) in the case of the multiple bang–bang control of the present model.

The amplitude of the tip of the beam in Fig. 5 is seen to be almost the same as that in Fig. 6. It is shown that the final times are almost equal both for the multiple bang–bang control with $t_f = 2.385$ s and the present new analysis with $t_f = 2.39$ s obtained by the minimum-energy, minimum-time optimized maneuver.

Note that the new control profile of the singular mode suffices for implementation of an analog actuator such as a continuous actuator, whereas multiple bang–bang control is appropriate for the implementation of an on–off actuator such as pulse jets.

The proposed profile is seen to have a remarkable feature in the amount of fuel usage. The control profile of the present analysis minimizes the total energy of the system in the rest-to-rest maneuver and, thus, equivalently minimizes the amount of fuel usage in the system. Figure 7 shows a comparison of the time histories of fuel usage:

$$F_{\text{usage}}(t) = \int_0^t |u(\tau)|\,\mathrm{d}\tau \tag{28}$$

between the minimum-energy, minimum-time optimized maneuver and the multiple bang–bang control of the time-optimal control.

Fuel usage of the minimum-energy, minimum-time optimized maneuver is seen to decrease in comparison with that of the time-optimal control.

VI. Conclusions

An algorithm to maneuver flexible space structures from a rest state to another rest state is studied to move the system with minimal excitation of flexible vibration and in minimum time. The problem is then studied for the minimum-energy maneuver with consideration of the total energy induced of the system and the minimization of maneuver time. The problem of the minimum-energy maneuver is a singular optimal control problem, and functional ideas are used to reduce the control problem to consideration of certain integral equations. The optimal control profile for minimum-energy control is obtained in an analytic manner. The number of integral equations to be solved is equal to the number of controllers in the analysis, and the present method is appropriate to treat flexible space structures, which usually have much smaller numbers of controllers than that of flexible vibration modes to be controlled.

Time-optimized control is studied numerically as the least-time maneuver of the energy-minimum maneuver that does not violate the constraint on the control input. The present method of functional analysis has the advantage that an analytical solution is attainable, and this leads to a number of possibilities to improve the performance of the maneuver by applying numerical procedure on the analytical solution. This advantage of the present approach is applied to analyze numerically the time-optimal performance. A simple model of flexible space structures is treated to study the time-optimal performance of the present energy-minimum maneuver. The resulting time-optimal minimum-energy maneuver is compared in its performance with the well-known time-optimal control, the switched bang–bang control, for excitation of vibration on a flexible structure, time necessary for the maneuver, and fuel consumption. For the time-optimal minimum-energy maneuver, the present method is shown have advantages over bang–bang control. In addition, it has advantages for the appropriate application to flexible space structures because of the reduction of control efforts by employing continuous controlled jets with a little sacrifice of the maneuver time.

Appendix: Proof of Convexity of the Lagrangian

To simplify the proof of the convexity of the Lagrangian, the number of the control inputs is assumed to be 1. The general case can be shown in the same manner.

The Lagrangian is defined as

$$I(u,\lambda) = \nu^T[Lu(t_f) - x_f] + \tfrac{1}{2}\langle Lu, QLu\rangle_1 - \langle\lambda, \varphi(u)\rangle_2$$
$$\varphi(u) = 1 - u^2 \geq 0$$

It is sufficient to show that

$$\alpha I(u,\lambda) + \beta I(v,\lambda) - I(\alpha u + \beta v) \geq 0$$

for arbitrary $u, v \in H_2$ and $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$. Thus,

$$\begin{aligned}
&\alpha I(u,\lambda) + \beta I(v,\lambda) - I(\alpha u + \beta v)\\
&\quad = \alpha\nu^T[Lu(t_f) - x_f] + \beta\nu^T[Lv(t_f) - x_f]\\
&\qquad - \nu^T[L(\alpha u + \beta v)(t_f) - x_f] + \tfrac{1}{2}\alpha\langle Lu, QLu\rangle_1\\
&\qquad + \tfrac{1}{2}\beta\langle Lv, QLv\rangle_1 - \tfrac{1}{2}\langle L(\alpha u + \beta v), QL(\alpha u + \beta v)\rangle_1\\
&\qquad - \alpha\langle\lambda, \varphi(u)\rangle_2 - \beta\langle\lambda, \varphi(v)\rangle_2 + \langle\lambda, \varphi(\alpha u + \beta v)\rangle_2\\
&\quad = \nu^T[\alpha Lu(t_f) - \alpha x_f + \beta Lv(t_f) - \beta x_f - \alpha Lu(t_f)\\
&\qquad - \beta Lv(t_f) + x_f] + \tfrac{1}{2}\alpha\langle Lu, QLu\rangle_1 + \tfrac{1}{2}\beta\langle Lv, QLv\rangle_1\\
&\qquad - \tfrac{1}{2}\alpha^2\langle Lu, QLu\rangle_1 - \alpha\beta\langle Lu, QLv\rangle_1 - \tfrac{1}{2}\beta^2\langle Lv, QLv\rangle_1\\
&\qquad - \langle\lambda, \alpha(1 - u^2) + \beta(1 - v^2) - \{1 - (\alpha u + \beta v)^2\}\rangle_2\\
&\quad = \nu^T[(-\alpha - \beta + 1)x_f]\\
&\qquad + \tfrac{1}{2}\alpha(1-\alpha)\langle Lu, QLu\rangle_1 + \tfrac{1}{2}\beta(1-\beta)\langle Lv, QLv\rangle_1\\
&\qquad - \alpha\beta\langle Lu, QLv\rangle_1 - \langle\lambda, \alpha - \alpha u^2 + \beta - \beta v^2\\
&\qquad - \{1 - \alpha^2u^2 - 2\alpha\beta uv - \beta^2v^2\}\rangle_2\\
&\quad = \tfrac{1}{2}\alpha\beta\langle Lu, QL(u - v)\rangle_1 - \tfrac{1}{2}\alpha\beta\langle L(u - v), QLv\rangle_1\\
&\qquad - \langle\lambda, -\alpha(1-\alpha)u^2 + 2\alpha\beta uv - \beta(1-\beta)v^2\rangle_2\\
&\quad = \tfrac{1}{2}\alpha\beta\langle Lu, QL(u - v)\rangle_1 - \tfrac{1}{2}\alpha\beta\langle Lv, QL(u - v)\rangle_1\\
&\qquad + \alpha\beta\langle\lambda, u^2 - 2uv + v^2\rangle_2\\
&\quad = \tfrac{1}{2}\alpha\beta\langle L(u - v), QL(u - v)\rangle_1 + \alpha\beta\langle\lambda, (u - v)^2\rangle_2 \geq 0 \qquad \square
\end{aligned}$$

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